

The derivation of model kinetic equation for gases and for plasmas

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A new form of the model collision operator for a Boltzmann gas of hard spheres and Coulomb plasma is derived. One-component and many-component systems are considered. The collision operator proposed takes properly into account the relaxation of the first 13 hydrodynamic moments. An expression for the intensity of the Langevin source in the model kinetic equation is obtained in the same approximation. A technique for reconstruction of the model collision integral based on a known expression for the model linearized operator is proposed. It is shown that, within our model, the collision integral does not contain a complicated exponential, common for the ellipsoidal statistical type models. Boltzmann's H-theorem is proved for our model.

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I. INTRODUCTION

As it is well known, neither the Boltzmann kinetic equation for gases nor the Landau or Balescu-Lenard equations for a plasma can be resolved exactly and one uses approximations and models that preserve the essential properties of the original collision operator. The most widely used model kinetic equation, especially efficient in the case of discrete simulation, for example in the lattice Boltzmann gas calculations [1], is the Bhatnagar, Gross and Krook (BGK) model [2]. We recall that in the BGK model the collision term for the one-component system

$$I^{BGK}\{f\} = -\nu(f - f^0) \quad (1)$$

is the deviation of the distribution function (d.f.) f from the Maxwellian d.f.

$$f^0 = \frac{n}{(2\pi mT)^{3/2}} \exp -\frac{m(\mathbf{v} - \mathbf{V})^2}{2T} \quad (2)$$

whose parameters: $n(\mathbf{r}, t) = \int f d\mathbf{p}$; $\mathbf{V}(\mathbf{r}, t) = \frac{1}{n} \int \mathbf{v} f d\mathbf{p}$; $\mathbf{T}(\mathbf{r}, t) = \int \frac{m(\mathbf{v} - \mathbf{V})^2}{3n} f d\mathbf{p}$ - the local density, the mean velocity and the temperature in energy units, are, respectively, moments of the distribution function f . The term (1) vanishes at equilibrium and satisfies the conservation laws:

$$\int \varphi(\mathbf{p}) I^{BGK}\{f\} d\mathbf{p} = 0, \text{ if } \varphi(\mathbf{p}) = 1, \mathbf{p}, \frac{\mathbf{p}^2}{2m} \quad (3)$$

and Boltzmann's H-theorem [3]:

$$\frac{\partial}{\partial t} H^{BGK}(t) = \nu \int (f - f^0) \log \frac{f}{f^0} d\mathbf{p} \leq 0. \quad (4)$$

In the problems of linear transport and fluctuations, one commonly uses the linearized form of the BGK collision operator:

$$\delta \hat{I}|h\rangle = -\nu \left(|h\rangle - \sum_{\alpha=1}^5 |\Psi_\alpha\rangle \langle \Psi_\alpha|h\rangle \right), \quad (5)$$

where $|h\rangle$ is defined by $f = f^0 + \delta f = f^0 (1 + h)$, and $|\Psi_\alpha\rangle$ are the first five Hermite polynomials.

The advantage of the BGK model is that the solution of the kinetic equation reduces to that of a system of algebraic equations [3]. A weak point is that the model implies that the Prandtl number (Pr) equals 1, while for monatomic gases the Prandtl number amounts to 2/3.

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Holway [4] introduced the ellipsoidal statistical model in order to take into account real Prandtl numbers by substituting the local anisotropic Gaussian distribution for the local Maxwellian distribution:

$$f^0 = n\pi^{-3/2}(\det A)^{1/2} \exp - \sum_{i,j=1}^3 \alpha_{ij}(\mathbf{v}_i - \mathbf{V}_i)(\mathbf{v}_j - \mathbf{V}_j);$$

$$A = \|\alpha_{ij}\| = \|(\text{Pr})^{-1}(2T/m)\delta_{ij} - 2(1 - \text{Pr})P_{ij}/n\text{Pr}\|^{-1}. \quad (6)$$

$$P_{ij} = m \int d\mathbf{p} f[(\mathbf{v}_i - \mathbf{V}_i)(\mathbf{v}_j - \mathbf{V}_j) - \delta_{ij} \frac{(\mathbf{v} - \mathbf{V})^2}{3}]$$

is the pressure tensor.

Similar problems arise in the calculation of fluctuation characteristics in gases and plasmas. As it is well known, the fluctuations of the distribution function can be determined by a fluctuation kinetic equation of Boltzmann type (for neutral particles) or Landau or Balescu-Lenard type (for plasma) with additional random source, called by analogy with Brownian motion the "Langevin source". The intensity of this Langevin source in an equilibrium one-component system is determined by the linearized collision operator

$$(yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} = -(\delta \hat{I}_{\mathbf{p}_1} + \delta \hat{I}_{\mathbf{p}_2})f^0(\mathbf{p}_1)\delta(\mathbf{p}_1 - \mathbf{p}_2). \quad (7)$$

The kinetic theory of such fluctuations for Boltzmann gases was developed for the first time by Kadomtsev [5]. The most complete survey of studies for nonequilibrium electron gases was given in [6, 7]. The theory of kinetic fluctuations in nonequilibrium multicomponent gases and plasmas was developed by Klimontovich [8]. Such a Langevin approach, which is widely used in different fields, is very convenient for calculating fluctuation characteristics. For example, the spectral function of the fluctuations of distribution function can be expressed in terms of the intensity of the Langevin source (7) and Green's function of the linearized kinetic equation. The intensity of the Langevin source must be found in the same approximation as the solution of the linearized kinetic equation with collisions. In the BGK approximation, the intensity of the Langevin source has the form:

$$(yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} = 2\nu f^0(\mathbf{p}_1)\{\delta(\mathbf{p}_1 - \mathbf{p}_2) - f^0(\mathbf{p}_2)[1 + \frac{m\delta\mathbf{v}_1\delta\mathbf{v}_2}{T} + \frac{1}{6}\left(3 - \frac{m\delta\mathbf{v}_1^2}{T}\right)\left(3 - \frac{m\delta\mathbf{v}_2^2}{T}\right)]\}; \quad \delta\mathbf{v} = \mathbf{v} - \mathbf{V}. \quad (8)$$

The expression (8) has the invariant properties

$$\int \Phi(\mathbf{p}_1)\Psi(\mathbf{p}_2)(yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} d\mathbf{p}_1 d\mathbf{p}_2 = 0 \quad (9)$$

for Φ or $\Psi = 1, \mathbf{p}, \mathbf{p}^2/2m$, but it does not give correct values for the intensities of Landau-Lifshitz Langevin sources in hydrodynamic equations [9]. In such a description, the intensities of the external stress tensor and the external heat flux vector are determined by the same relaxation frequency ν .

A graver situation arises in the case of many-component systems. According to the Gross and Krook (GK) model [10], the collision operator has the form of the deviation of d.f. from a "mythical" exponent:

$$I_a^{GK}\{f_a\} = - \sum_b \nu_{ab} [f_a - \frac{n_a}{(2\pi m_a T_{ab})^{3/2}} \exp - \frac{m_a(\mathbf{v} - \mathbf{V}_{ab})^2}{2T_{ab}}], \quad (10)$$

where the parameters \mathbf{V}_{ab} and T_{ab} are related linearly to the d.f. moments $\mathbf{V}_a; \mathbf{V}_b; T_a; T_b$:

$$\mathbf{V}_{ab} = \alpha_{aa}\mathbf{V}_a + \alpha_{ab}\mathbf{V}_b, \quad T_{ab} = \beta_{aa}T_a + \beta_{ab}T_b. \quad (11)$$

Coefficients $\alpha_{aa}, \alpha_{ab}, \beta_{aa}$ and β_{ab} are chosen in such manner that both the conservation laws and balance equations for the momenta and energy for each component hold valid. Since the number of equations to be satisfied by the parameters of the model (for the five-moment description of a two-component system there are four equations: two

for the balance of moments and two for the balance of temperature) is less than the number of unknown parameters (in this approximation these are five: ν_{ab} ; α_{aa} ; α_{ab} ; β_{aa} ; β_{ab}), there is an arbitrariness in the choice of parameters. Therefore there exist various modifications (see, for example, [11]) of the collision model, which correctly describe relaxation of the five moments. But, probably, the most dubious point of the GK model is the complicated exponential dependence on d.f. Recently, a new type of G-K, so-called ellipsoidal statistical model for gas mixtures [12–14], was proposed ad hoc and the Boltzmann’s H- theorem was proven for this model. However, the conservation laws and the H-theorem are but a necessary, and not a sufficient conditions for a model to be satisfactory. The correct model should be derived. One of the first works dedicated to derivation of the model linearized collision integral is the paper by Gross and Jackson (GJ) [15]. Later, extension of the BGK kinetic model for the inclusion of higher order matrix elements was discussed and it was applied to investigate the generalized Enskog equation and the dynamic structure factor for gas and fluids [16–20]. The approximation consisted in taking into account exactly a finite part of the matrix operator, while the remaining part was only represented by the diagonal matrix elements. In the present paper we do not consider spatial inhomogeneities and assume the wave vector $\mathbf{k} = 0$. But we do take into account the non-diagonal components arising in the collision operator expansion with respect to the complete system of polynomials in the quadratic approximation. Making use of these non-diagonal elements allows us to obtain a new form for the model integral for Coulomb plasma with transport coefficient correction, comparable with the Spitzer corrections. In the case of a Boltzmann gas of hard spheres these corrections are insignificant. An expression for the intensity of the Langevin source in the model kinetic equation is obtained in the same approximation. From this expression follow correct values for the intensities of Landau-Lifshitz Langevin sources in hydrodynamic equations. Using the technique developed for an one-component system, a consistent derivation of the model linearized collision operator for a many-component system is presented. In these results an ambiguity in the choice of coefficients is eliminated, in contrast to the GK type models. A technique for reconstruction of the form of the model collision integral based on a known expression for the model linearized operator is proposed. It is shown that the model collision integral in the local (not complete) equilibrium approximation does not contain a complicated exponential, that is common for the GK type integrals. Boltzmann’s H-theorem is proved for our model.

II. MODEL CONSTRUCTION

A. One-component systems

To correct the BGK model following Sirovich [21] we introduce two projection operators \hat{H} and \hat{N} satisfying

$$\hat{H}\hat{N} = \hat{N}\hat{H} = 0; \quad \hat{H} + \hat{N} = \hat{I}d. \quad (12)$$

Here $\hat{I}d$ is the identity operator, \hat{H} is the operator of projection onto the ‘hydrodynamic subspace’ spanned by kets corresponding to the polynomials of the lowest order in the moment variable. In the BGK model these kets are the first five polynomials which correspond to the collision invariants: density, momentum and kinetic energy. However, one may include higher-order polynomials in this subspace. Their number and order depend on the physical processes that one wishes to treat ‘‘exactly’’. Thus, one may take into account non-invariant values like the pressure tensor and heat flux. The projection operator \hat{N} maps the state vector onto the remaining ‘‘non-hydrodynamic subspace’’. Since we are interested in a model operator describing the first 13 moments correctly we take the operator \hat{H} in the following form:

$$\hat{H} = \sum_{i=1}^{13} |\Psi_i\rangle \langle \Psi_i|, \quad (13)$$

where the first 13 Hermite polynomials in a Cartesian coordinate system are [22]:

$$\begin{aligned} |\Psi_1\rangle &= 1; |\Psi_{r+1}\rangle = u_r, r = 1, 2, 3; |\Psi_5\rangle = \sqrt{1/6}(u^2 - 3); |\Psi_6\rangle = \sqrt{3/2}(u_x^2 - \frac{1}{3}u^2); |\Psi_7\rangle = 1/2(u_y^2 - u_z^2); |\Psi_8\rangle = u_x u_z; \\ |\Psi_9\rangle &= u_x u_y; |\Psi_{10}\rangle = u_y u_z; |\Psi_{r+10}\rangle = \sqrt{1/10}u_r(u^2 - 5), r = 1, 2, 3, \\ \mathbf{u} &= (\mathbf{p} - m\mathbf{V})/(mT)^{1/2} \text{ is the dimensionless velocity.} \end{aligned}$$

The linearized collision operator is:

$$\delta\hat{I} = \hat{H}\delta\hat{I}\hat{H} + \hat{H}\delta\hat{I}\hat{N} + \hat{N}\delta\hat{I}\hat{H} + \hat{N}\delta\hat{I}\hat{N}. \quad (14)$$

Since the first five Hermite polynomials are the eigenfunctions of the collision operator for identical particles corresponding to the zero eigenvalue, it follows that for $1 \leq i \leq 5$ one has $\hat{H}\delta\hat{I}\hat{H} = \hat{H}\delta\hat{I}\hat{N} = \hat{N}\delta\hat{I}\hat{H} = 0$. The higher

Hermite polynomials are eigenfunctions of the collision operator only for a Maxwell's molecule with a model repulsive potential proportional to r^{-4} . In this case, non-diagonal matrix elements $\hat{H}\delta\hat{I}\hat{N}$ and $\hat{N}\delta\hat{I}\hat{H}$ vanish:

$$\hat{H}\delta\hat{I}\hat{N} = \hat{N}\delta\hat{I}\hat{H} = 0. \quad (15)$$

For any other interaction potentials the Hermite polynomials are not the eigenfunctions of the collision operator, the equality (15) does not hold and the collision operator matrix elements contain non-diagonal elements. Our first approximation is that we accept (15) as a valid formula for the Boltzmann gas of hard spheres and Coulomb plasma. However, the first approximation is not sufficient for describing real gas and plasma. In the second approximation we take into account only the non-diagonal terms closest to the diagonal. As we will show below, the corrections for a Boltzmann gas of hard spheres turns out to be small, but for Coulomb systems they are not small and comparable with the Spitzer corrections to transport coefficients. We can continue this process and take into account in the third approximation the next, non-diagonal, terms more distant from the diagonal elements. We performed these calculations and found that the third approximation yields very small corrections (compared to the second approximation), that can be neglected.

Since for one-component systems the operator is Hermitian and isotropic, Wigner-Ekkart theorem [23] implies and the selection rule follows: the contribution to the non-diagonal matrix elements $\hat{H}\delta\hat{I}\hat{N}$ and $\hat{N}\delta\hat{I}\hat{H}$ is given only by polynomials with identical pairs of orbital numbers. For example, for the polynomial $|\Psi_6\rangle = \frac{\sqrt{3}}{2}(u_x u_x - \frac{1}{3}u^2)$ defining the xx component of the pressure tensor, the non-zero contribution to the non-diagonal matrix elements is given by non-hydrodynamic polynomials of higher order in u^2 but with the same values of l and m ($l = 2; m = 2$). $\Psi_6^{(2)} = \sqrt{\frac{3}{14}}\frac{1}{2}(u^2 - 7)(u_x u_x - \frac{1}{3}u^2)$.

The main modeling procedure consists of approximating the non-hydrodynamic contribution. If the operator \hat{H} involves the first 13 Hermite polynomials, then the neglect of the term $\hat{N}\delta\hat{I}\hat{N}$ does not affect calculations for such transport coefficients as viscosity and heat conductivity. Nevertheless the approximation

$$\hat{N}\delta\hat{I}\hat{N} = -\nu\hat{N} \quad (16)$$

which is reduced to the partial breach of the fine structure of its spectrum by the contraction of all the eigenvalues of the \hat{N} to the minimum value, allows one to describe at least qualitatively the 'tails' of neglected 'non-hydrodynamic' terms (ν corresponds to the longest non-hydrodynamic relaxation time). An account of these 'tails' may be important at the kinetic level of fluctuation description. Using this approximation one may rewrite, in a first approximation, that corresponds to the Maxwell's molecule, the model operator as follows:

$$\delta\hat{I} = -\nu\hat{I}d + \hat{H}(\delta\hat{I} + \nu)\hat{H}. \quad (17)$$

For the 13 moment basis for \hat{H} in the first approximation one has

$$\delta\hat{I}|h\rangle = -\nu|h\rangle + \nu \sum_{i=1}^5 |\Psi_i\rangle \langle \Psi_i|h\rangle + \sum_{i=6}^{13} |\Psi_i\rangle (\langle \Psi_i|\delta\hat{I}|\Psi_i\rangle + \nu) \langle \Psi_i|h\rangle. \quad (18)$$

In the same approximation, the expression for the intensity of the Langevin source has the form:

$$(yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} = 2f^0(\mathbf{p}_1)[\nu\delta(\mathbf{p}_1 - \mathbf{p}_2) - \nu f^0(\mathbf{p}_2) \sum_{i=1}^5 \Psi_i(\mathbf{u}_1)\Psi_i(\mathbf{u}_2) + f^0(\mathbf{p}_2) \sum_{i=6}^{13} \Psi_i(\mathbf{u}_1)\Psi_i(\mathbf{u}_2)(\langle \Psi_i|\delta\hat{I}|\Psi_i\rangle + \nu)] \quad (19)$$

The first two terms in (19) correspond to the BGK model (8).

In the second approximation, the nearest non-diagonal entries appear in (18):

$$\sum_{i=6}^{13} |\Psi_i\rangle \langle \Psi_i|\delta\hat{I}|\Psi_i^{(2)}\rangle \langle \Psi_i^{(2)}|h\rangle, \quad (20)$$

where the non-hydrodynamic polynomials, which we take into account, are

$$\Psi_i^{(2)}(u) = \frac{1}{\sqrt{14}}(u^2 - 7)\Psi_i(u), \quad 6 \leq i \leq 10 \quad (21)$$

$$\Psi_{r+10}^{(2)}(u) = \frac{1}{\sqrt{280}}(u^4 - 14u^2 + 35)\Psi_r(u), \quad 1 \leq r \leq 3. \quad (22)$$

The non-hydrodynamical moments of polynomials (21, 22) are determined in the Fourier presentation and $k = 0$ by the following equations

$$(-i\omega - \langle \Psi_i^{(2)} | \delta \hat{I} | \Psi_i^{(2)} \rangle) \langle \Psi_i^{(2)} | h \rangle_\omega = \langle \Psi_i^{(2)} | \delta \hat{I} | \Psi_i \rangle \langle \Psi_i | h \rangle_\omega. \quad (23)$$

Thus in the second approximation, the linearized model collision operator has the form

$$\delta \hat{I} | h \rangle_\omega = -\nu | h \rangle_\omega + \nu \sum_{i=1}^5 | \Psi_i \rangle \langle \Psi_i | h \rangle_\omega - \sum_{i=6}^{13} | \Psi_i \rangle ({}_i\Lambda_i^{(2)}(\omega) - \nu) \langle \Psi_i | h \rangle_\omega, \quad (24)$$

where

$${}_i\Lambda_i^{(2)}(\omega) = -\langle \Psi_i | \delta \hat{I} | \Psi_i \rangle - \frac{\langle \Psi_i | \delta \hat{I} | \Psi_i^{(2)} \rangle^2}{-i\omega - \langle \Psi_i^{(2)} | \delta \hat{I} | \Psi_i^{(2)} \rangle} \quad (25)$$

contains the square of the non-diagonal entries and the projection of the resolvent of the kinetic equation onto the non-hydrodynamical subspace. Here we take into account non-stationarity of non-hydrodynamical moments. Thus, although the original collision integral is Markov, the part projected onto the subspace of the 13 moments becomes, in the second approximation, a frequency dependent operator. A similar situation occurs in quantum-mechanical perturbation theory. Note that in the Markov approximation, the second order corrections in (25) are negative for any interaction potentials. Calculate now matrix elements of the operator for a special interaction potential, namely for the Coulomb plasma and the Boltzmann hard sphere gas.

1. Coulomb One-Component Plasma

For the Coulomb one-component plasma, the time evolution of the fluctuations of the particle distribution function is determined by the equation

$$\frac{\partial}{\partial t} | h \rangle + \hat{\Phi} | h \rangle + \hat{V} | h \rangle - \delta \hat{I} | h \rangle = | y \rangle, \quad (26)$$

where $\hat{\Phi}$ is the operator that describes the free motion, and \hat{V} is the linearized Vlasov operator. The intensity of the Langevin source $| y \rangle$ in (26) differs from the expressions (8) and (19) by the quantity $[1/f^0(\mathbf{p}_1)f^0(\mathbf{p}_2)]$. $\delta \hat{I}$ is the linearized collision operator, its action being determined by [24]

$$\delta \hat{I} | h \rangle = \frac{1}{f^0} \int \left(\frac{\partial}{\partial p_r} - \frac{\partial}{\partial p_r'} \right) Q^{rs} \left(\frac{\partial}{\partial p_s} - \frac{\partial}{\partial p_s'} \right) f^0(\mathbf{p}) f^0(\mathbf{p}') (h(\mathbf{p}) + h(\mathbf{p}')) d\mathbf{p}', \quad (27)$$

where the kernel Q^{rs} in the Balescu-Lenard form is

$$Q^{rs} = 2e^4 \int d\mathbf{k} \frac{k_r k_s}{k^4 |\varepsilon(\mathbf{k}\mathbf{v}, \mathbf{k})|^2} \delta(\mathbf{k}\mathbf{v} - \mathbf{k}\mathbf{v}').$$

The matrix element of the operator (27) can be represented in the form

$$\langle \Psi_i | \delta \hat{I} | \Psi_j \rangle = -\frac{1}{2} \int Q^{rs} f^0(\mathbf{p}) f^0(\mathbf{p}') \left(\frac{\partial \Psi_i}{\partial \mathbf{p}_r} - \frac{\partial \Psi_i}{\partial \mathbf{p}_r'} \right) \left(\frac{\partial \Psi_j}{\partial \mathbf{p}_s} - \frac{\partial \Psi_j}{\partial \mathbf{p}_s'} \right) d\mathbf{p} d\mathbf{p}'. \quad (28)$$

For $i = j$, $\langle \Psi_i | \delta \hat{I} | \Psi_j \rangle \leq 0$. Vanishing the matrix elements $\langle \Psi_i | \delta \hat{I} | \Psi_j \rangle$ (for $i = 1 - 5$) corresponds to the fivefold degenerate zero eigenvalue of the operator $\delta \hat{I}$. It is easy to show that the matrix elements of $\hat{H} \delta \hat{I} \hat{H}$ have the values

$$\langle \Psi_i | \delta \hat{I} | \Psi_j \rangle = -\delta_{ij} \left[\Lambda_1 \sum_{k=6}^{10} \delta_{ik} + \Lambda_2 \sum_{k=11}^{13} \delta_{ik} \right], \quad (29)$$

where Λ_1, Λ_2 are the reciprocal relaxation times of the pressure tensor and the heat flux vector which in terms of the plasma parameters are as follows:

$$\Lambda_1 = \frac{1}{10} \frac{\lambda}{\pi^{3/2}} \omega_L \ln \frac{1}{\lambda}; \quad \Lambda_2 = \frac{2}{3} \Lambda_1, \quad (30)$$

where $\omega_L = \left(\frac{4\pi n e^2}{m} \right)^{1/2}$ is the electron plasma frequency, and $\lambda = n^{-1} k_D^3$ is the plasma parameter.

The matrix elements of the operators $\hat{N} \delta \hat{I} \hat{N}$ and $\hat{H} \delta \hat{I} \hat{N}$ defining 'tails' and the corrections of the second approximation, have for a Coulomb plasma the values [25]

$$\begin{aligned} \langle \Psi_{14} | \delta \hat{I} | \Psi_{14} \rangle &= -\frac{2}{3} \Lambda_1; \quad \Psi_{14} = \frac{1}{\sqrt{120}} (u^4 - 10u^2 + 15), \\ \langle \Psi_{15} | \delta \hat{I} | \Psi_{15} \rangle &= \dots = \langle \Psi_{21} | \delta \hat{I} | \Psi_{21} \rangle = -\frac{3}{2} \Lambda_1; \quad \Psi_{15} = u_x u_y u_z, \\ \langle \Psi_{22} | \delta \hat{I} | \Psi_{22} \rangle &= \dots = \langle \Psi_{35} | \delta \hat{I} | \Psi_{35} \rangle = -\frac{191}{16} \Lambda_1; \quad \Psi_{22} = \frac{1}{\sqrt{105}} \frac{1}{8} (35u_x^4 - 30u^2 u_x^2 + 3u^4), \\ \langle \Psi_i^{(2)} | \delta \hat{I} | \Psi_i^{(2)} \rangle &= -\frac{201}{168} \Lambda_1; \quad \langle \Psi_i | \delta \hat{I} | \Psi_i^{(2)} \rangle = \frac{3}{2\sqrt{14}} \Lambda_1; \quad 6 \leq i \leq 10, \\ \langle \Psi_{10+r}^{(2)} | \delta \hat{I} | \Psi_{10+r'}^{(2)} \rangle &= -\frac{15}{14} \delta_{rr'} \Lambda_1, \\ \langle \Psi_{10+r} | \delta \hat{I} | \Psi_{10+r'}^{(2)} \rangle &= \frac{1}{7} \delta_{rr'} \Lambda_1; \quad 1 \leq r \leq 3. \end{aligned}$$

This estimate implies that the higher tensor character of the polynomial leads to higher values of the matrix elements and for polynomials with the same tensor character the matrix elements are greater for polynomials of higher order in u^2 . The smallest value of the diagonal matrix elements for the non-hydrodynamical polynomial is achieved for the Ψ_{14} polynomial and equals Λ_2 (the heat flux relaxation frequency). The same holds for Maxwell's molecule and the Boltzmann gas hard sphere at least. Thus in the first approximation the linearized collision operator and the intensity of the Langevin source have the form:

$$\delta \hat{I} | h \rangle = -\frac{2}{3} \Lambda_1 | h \rangle + \frac{2}{3} \Lambda_1 \sum_{i=1}^5 |\Psi_i\rangle \langle \Psi_i | h \rangle - \frac{1}{3} \Lambda_1 \sum_{i=6}^{10} |\Psi_i\rangle \langle \Psi_i | h \rangle, \quad (31)$$

$$(yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} = \frac{4}{3} \Lambda_1 f^0(\mathbf{p}_1) [\delta(\mathbf{p}_1 - \mathbf{p}_2) - f^0(\mathbf{p}_2) \sum_{i=1}^5 \Psi_i(\mathbf{u}_1) \Psi_i(\mathbf{u}_2) + \frac{1}{2} f^0(\mathbf{p}_2) \sum_{i=6}^{13} \Psi_i(\mathbf{u}_1) \Psi_i(\mathbf{u}_2)]. \quad (32)$$

We see that the polynomials corresponding to the heat flux disappear. It is easy to show that (31) and (32) gives the well-known values of the transport coefficients in the hydrodynamic equations and the Landau-Lifshitz formulas [9].

In the second approximation it holds

$$\delta \hat{I} | h \rangle_{\omega} = -\nu(|h\rangle_{\omega}) - \sum_{i=1}^{13} |\Psi_i\rangle \langle \Psi_i | h \rangle_{\omega} - \sum_{i=6}^{10} \Lambda_1^{(2)}(\omega) |\Psi_i\rangle \langle \Psi_i | h \rangle_{\omega} - \sum_{i=11}^{13} \Lambda_2^{(2)}(\omega) |\Psi_i\rangle \langle \Psi_i | h \rangle_{\omega}, \quad (33)$$

and

$$\begin{aligned} (yy)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} &= 2f^0(\mathbf{p}_1) [\nu \delta(\mathbf{p}_1 - \mathbf{p}_2) - \nu f^0(\mathbf{p}_2) \sum_{i=1}^5 \Psi_i(\mathbf{u}_1) \Psi_i(\mathbf{u}_2) \\ &+ \text{Re} \Lambda_1^{(2)}(\omega) f^0(\mathbf{p}_2) \sum_{i=6}^{10} \Psi_i(\mathbf{u}_1) \Psi_i(\mathbf{u}_2) + \text{Re} \Lambda_2^{(2)}(\omega) f^0(\mathbf{p}_2) \sum_{i=11}^{13} \Psi_i(\mathbf{u}_1) \Psi_i(\mathbf{u}_2)], \end{aligned} \quad (34)$$

where

$$\Lambda_1^{(2)}(\omega) = \Lambda_1 \left(1 - \frac{\Lambda_1 9/56}{-i\omega + \Lambda_1 205/168} \right);$$

$$\Lambda_2^{(2)}(\omega) = \Lambda_2(1 - \frac{\Lambda_1 3/14}{-i\omega + \Lambda_1 15/14}). \quad (35)$$

In the Markov approximation ($\omega = 0$)

$$\Lambda_1^{(2)} = \Lambda_1(1 - \frac{27}{205}); \quad \Lambda_2^{(2)} = \Lambda_2(1 - \frac{1}{5}). \quad (36)$$

the corrections are rather significant and comparable with the Spitzer corrections [26]. In the third approximation the relaxation frequencies vary by no more than one per cent.

From the scalar product of (26) with $|\Psi_k\rangle$ ($1 \leq k \leq 13$) there follows a system of equations for the hydrodynamic moments

$$\frac{\partial}{\partial t} \delta n + n \frac{\partial}{\partial \mathbf{q}} \delta \mathbf{V} = 0;$$

$$\frac{\partial}{\partial t} n \delta V_i = -\frac{1}{m} \frac{\partial \delta(nT)}{\partial q_i} - \frac{1}{m} \frac{\partial \delta P_{ij}}{\partial q_j} - \frac{1}{m} \delta E_i;$$

$$\frac{\partial}{\partial t} \delta T + \frac{2}{3} \frac{T}{n} \frac{\partial}{\partial \mathbf{q}} \delta \mathbf{V} + \frac{2}{3} \frac{1}{n} \frac{\partial}{\partial \mathbf{q}} \delta \mathbf{S} = 0; \quad (37)$$

$$\begin{aligned} \frac{\partial}{\partial t} \delta P_{ij} - nT \left(\frac{\partial \delta V_i}{\partial q_j} + \frac{\partial \delta V_j}{\partial q_i} - \frac{2}{3} \delta_{ij} \frac{\partial \delta V_k}{\partial q_k} \right) - \frac{2}{5} \left(\frac{\partial \delta S_i}{\partial q_j} + \frac{\partial \delta S_j}{\partial q_i} - \frac{2}{3} \delta_{ij} \frac{\partial \delta S_k}{\partial q_k} \right) \\ = -\Lambda_1^{(2)}(\omega) \delta P_{ij} + \frac{1}{m} \langle (p_i p_j - \frac{1}{3} \delta_{ij} p^2) | y \rangle; \end{aligned} \quad (38)$$

$$\frac{\partial}{\partial t} \delta S_i + \frac{5}{2} \frac{nT}{m} \frac{\partial \delta T}{\partial q_i} - \frac{T}{m} \frac{\partial \delta P_{ij}}{\partial q_j} = -\Lambda_2^{(2)}(\omega) \delta S_i + \frac{1}{2m^2} \langle p_i(p^2 - 5Tm) | y \rangle, \quad (39)$$

where

$$\begin{aligned} \delta P_{ij} = nT \{ \delta_{ix} \delta_{jx} \sqrt{\frac{4}{3}} \langle \Psi_6 | h \rangle + \delta_{iy} \delta_{jy} (\langle \Psi_7 | h \rangle) - \frac{1}{\sqrt{3}} \langle \Psi_6 | h \rangle - \delta_{iz} \delta_{jz} (\langle \Psi_7 | h \rangle) + \frac{1}{\sqrt{3}} \langle \Psi_6 | h \rangle + (\delta_{ix} \delta_{jz} + \delta_{iz} \delta_{jx}) \langle \Psi_8 | h \rangle \\ + (\delta_{ix} \delta_{jy} + \delta_{iy} \delta_{jx}) \langle \Psi_9 | h \rangle + (\delta_{iy} \delta_{jz} + \delta_{iz} \delta_{jy}) \langle \Psi_{10} | h \rangle \} \end{aligned} \quad (40)$$

is the fluctuation of the pressure tensor and

$$\delta S_r = \frac{mn}{2} \left(\frac{T}{2} \right)^{3/2} \sqrt{10} \langle \Psi_{10+r} | h \rangle \quad (41)$$

is the fluctuation of the heat flux.

In deriving (38) and (39), we have retained the terms with time derivatives, which for a plasma do not vanish in the long-wavelength approximation $k \rightarrow 0$. Making a Fourier transformation of (38), we obtain

$$\delta P_{ij}(\omega) = \delta \pi_{ij}(\omega) + \frac{1}{-i\omega + \Lambda_1^{(2)}(\omega)} \{ nT \left(\frac{\partial \delta V_i}{\partial q_j} + \frac{\partial \delta V_j}{\partial q_i} - \frac{2}{3} \delta_{ij} \frac{\partial \delta V_k}{\partial q_k} \right) + \frac{2}{5} \left(\frac{\partial \delta S_i}{\partial q_j} + \frac{\partial \delta S_j}{\partial q_i} - \frac{2}{3} \delta_{ij} \frac{\partial \delta S_k}{\partial q_k} \right) \}, \quad (42)$$

where

$$\delta\pi_{ij}(\omega) = \frac{1}{-i\omega + \Lambda_1^{(2)}(\omega)} \frac{1}{m} \langle (p_i p_j - \frac{1}{3} \delta_{ij} p^2) | y \rangle. \quad (43)$$

Similarly, it follows from (39) that

$$\delta S_i(\omega) = \sigma_i(\omega) - \frac{5}{2m} \frac{nT}{-i\omega + \Lambda_2^{(2)}(\omega)} \frac{\partial}{\partial q_i} \delta T(\omega), \quad (44)$$

where

$$\sigma_i(\omega) = -\frac{1}{-i\omega + \Lambda_2^{(2)}(\omega)} \frac{1}{2m^2} \langle p_i (p^2 - 5Tm) | y \rangle. \quad (45)$$

Using the expression for the spectral function of the Langevin source in the kinetic equation (34), we obtain

$$(\delta\pi_{ij}\delta\pi_{rl})_\omega = (\delta_{ir}\delta_{jl} + \delta_{il}\delta_{jr} - \frac{2}{3}\delta_{ij}\delta_{rl}) \frac{2nT^2 \text{Re}\Lambda_1^{(2)}(\omega)}{\left| -i\omega + \Lambda_1^{(2)}(\omega) \right|^2}. \quad (46)$$

Introducing the frequency-dependent viscosity

$$\eta(\omega) = \frac{nT}{-i\omega + \Lambda_1^{(2)}(\omega)} \quad (47)$$

we can rewrite the expression (46) as

$$(\delta\pi_{ij}\delta\pi_{rl})_\omega = 2T \text{Re}\eta(\omega) (\delta_{ir}\delta_{jl} + \delta_{il}\delta_{jr} - \frac{2}{3}\delta_{ij}\delta_{rl}). \quad (48)$$

Similarly,

$$(\sigma_r\sigma_l)_\omega = 2 \frac{T^2}{\kappa} \delta_{rl} \text{Re}\lambda(\omega), \quad (49)$$

with frequency-dependent thermal conductivity

$$\lambda(\omega) = \frac{5}{2} \frac{nT\kappa}{m} \frac{1}{-i\omega + \Lambda_2^{(2)}(\omega)}. \quad (50)$$

In the low-frequency limit $\omega \ll \Lambda_1, \Lambda_2$ Eqs. (48, 49) yield the classical results [9]. In the first approximation ($\Lambda_{1,2}^{(2)}(\omega) = \Lambda_{1,2}$), the results (48, 49) go over into the results obtained earlier [27]. In the general case, when the non-Markov processes are important, the transport coefficients (47) and (50) that occur in the expressions (48) and (49) contain continued fractions.

2. Boltzmann Gas of Hard Spheres

For a Boltzmann gas of hard spheres, the linearized collision operator has the form

$$\delta\hat{I}|h\rangle = \frac{R^2}{2f^0(\mathbf{p})} \int |(\mathbf{V}\mathbf{e})| f^0(\mathbf{p}) f^0(\mathbf{p}_1) \times [h(\mathbf{p}_1') + h(\mathbf{p}') - h(\mathbf{p}_1) - h(\mathbf{p})] d\mathbf{p}_1 d\Omega, \quad (51)$$

where R is the interaction range, $\mathbf{V} = \mathbf{v} - \mathbf{v}_1$ is the relative velocity of the particles, $\mathbf{v}' = \mathbf{v} + \mathbf{e}(\mathbf{eV})$, $\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{e}(\mathbf{eV})$, \mathbf{e} is an arbitrary unit vector.

The matrix elements of the operator (51) have the form

$$\langle \Psi_i | \delta\hat{I} | \Psi_j \rangle = -\frac{R^2}{8n} \int |(\mathbf{V}\mathbf{e})| f^0(\mathbf{p}) f^0(\mathbf{p}') [\Psi_i(\mathbf{p}_1') + \Psi_i(\mathbf{p}') - \Psi_i(\mathbf{p}_1) - \Psi_i(\mathbf{p})] [\Psi_j(\mathbf{p}_1') + \Psi_j(\mathbf{p}') - \Psi_j(\mathbf{p}_1) - \Psi_j(\mathbf{p})] d\mathbf{p} d\mathbf{p}_1. \quad (52)$$

In the hydrodynamic space

$$\langle \Psi_i | \delta \hat{I} | \Psi_j \rangle = -\delta_{ij} [\Lambda_1^G \sum_{k=6}^{10} \delta_{ik} + \Lambda_2^G \sum_{k=11}^{13} \delta_{ik}], \quad (53)$$

where

$$\Lambda_1^G = \frac{16}{5} \sqrt{\pi n R^2} \sqrt{\frac{T}{m}}, \quad \Lambda_2^G = \frac{32}{15} \sqrt{\pi n R^2} \sqrt{\frac{T}{m}} \quad (54)$$

The matrix elements that determine the correction of the second approximation for the Boltzmann gas take the values

$$\begin{aligned} \langle \Psi_{14} | \delta \hat{I} | \Psi_{14} \rangle &= -\frac{2}{3} \Lambda_1^G; \quad \langle \Psi_{15} | \delta \hat{I} | \Psi_{15} \rangle = \dots = \langle \Psi_{21} | \delta \hat{I} | \Psi_{21} \rangle = -\frac{3}{2} \Lambda_1^G, \\ \langle \Psi_i^{(2)} | \delta \hat{I} | \Psi_i^{(2)} \rangle &= -\frac{17}{14} \Lambda_1^G; \quad \langle \Psi_i | \delta \hat{I} | \Psi_i^{(2)} \rangle = -\frac{1}{2\sqrt{14}} \Lambda_1^G; \quad 6 \leq i \leq 10, \\ \langle \Psi_{10+r}^{(2)} | \delta \hat{I} | \Psi_{10+r'}^{(2)} \rangle &= -\frac{15}{14} \delta_{rr'} \Lambda_1^G, \\ \langle \Psi_{10+r} | \delta \hat{I} | \Psi_{10+r'}^{(2)} \rangle &= \frac{1}{3\sqrt{7}} \delta_{rr'} \Lambda_1^G; \quad 1 \leq r \leq 3. \end{aligned}$$

The second-order corrections for Boltzmann gas are one order less than those for the Coulomb plasma and one may stop at the first approximation. Thus in the case of a Boltzmann gas of hard spheres the collision operator may be represented in the form [28]

$$I\{f\} = -\nu \{f - f^0 (1 - P_{ij} \frac{\delta v_i \delta v_j}{4PT} m)\}, \quad (55)$$

Here, we have used the fact that $\Lambda_2^G = 2/3 \Lambda_1^G = \nu$.

For arbitrary Prandtl number Pr our model takes the form:

$$I\{f\} = -\nu \left\{ f - f^0 \left[1 - \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{ij} \frac{\delta v_i \delta v_j}{2PT} m \right] \right\}, \quad (56)$$

In the equilibrium state, $P_{ij} = 0$ and $I\{f^0\} = 0$. In the equation for the heat flux only the first term in (56) contributes. The relaxation of the pressure tensor is determined by both the first and the last terms in this equation.

Now we will prove the H theorem for our model (56):

$$\frac{\partial}{\partial t} H(t) = \nu \int (f - \Phi) \log \frac{f}{\Phi} d\mathbf{p} + \nu \int (f - \Phi) \left\{ \log f^0 + \log \left[1 - \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{kl} \frac{\delta v_k \delta v_l}{2PT} m \right] \right\} d\mathbf{p}, \quad (57)$$

where $\Phi = f^0 \left[1 - \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{ij} \frac{\delta v_i \delta v_j}{2PT} m \right]$.

The first term in the brace in (57) vanishes, while the second term can be expanded as: $\log(1+x) = x - x^2/2 + x^3/3 \dots$; for $-1 < x \leq 1$. Here only the first term contributes after the integration. Therefore:

$$\frac{\partial}{\partial t} H(t) = \nu \int (f - \Phi) \log \frac{f}{\Phi} d\mathbf{p} - \nu \int d\mathbf{p} \left\{ f - f^0 \left[1 - \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{ij} \frac{\delta v_i \delta v_j}{2PT} m \right] \right\} \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{kl} \frac{\delta v_k \delta v_l}{2PT} m. \quad (58)$$

Taking into account that

$$\int f^0 (\delta v_i \delta v_j - \delta_{ij} \frac{\delta v^2}{3}) (\delta v_k \delta v_l - \delta_{kl} \frac{\delta v^2}{3}) d\mathbf{p} = n \frac{T^2}{m^2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl}),$$

we obtain the H theorem:

$$\frac{\partial}{\partial t} H(t) = \nu \int (f - \Phi) \log \frac{f}{\Phi} d\mathbf{p} - \frac{\nu}{2PT} \frac{1 - \text{Pr}}{\text{Pr}^2} P_{ij} P_{ji} \leq \nu \int (f - \Phi) \log \frac{f}{\Phi} d\mathbf{p} \leq 0. \quad (59)$$

Thus our collision integral in the form (55) possesses all necessary properties and is free from the drawbacks of the one-component model of the BGK model mentioned above. The linearized form of (55) is congruent with the linearized ellipsoidal statistical model [3, 4].

Earlier, another model correctly describing the viscosity and thermal conductivity relaxation was proposed ad hoc [29]:

$$I\{f\} = -\nu\{f - f^0[1 - \text{Pr} m \frac{\mathbf{S}\delta\mathbf{v}}{PT}(\frac{m\delta v^2}{5T} - 1)]\}, \quad (60)$$

where

$$\mathbf{S} = m \int d\mathbf{p} \delta\mathbf{v} \frac{\delta v^2}{2} f$$

is the heat flux. But this model does not give a correct description of non-hydrodynamic 'tails'

B. Many-component systems

When the local equilibrium state is achieved, in the many-component systems, the stage of relaxation of the mean velocities and temperatures ensues and we can to restrict the hydrodynamical subspace to the first five polynomials. Using the technique described above for the one-component system, one may obtain the following expression for the linearized model of a many-component system in the five-moment approximation:

$$\delta\hat{I}_{a\mathbf{p}}\delta f_a(\mathbf{p}) = -\nu_a\delta f_a(\mathbf{p}) + \sum_{j=1}^5 \nu_a f_a^0(\mathbf{p}) \Psi_j^a(\mathbf{p}) \int \Psi_j^a(\mathbf{p}') \delta f_a(\mathbf{p}') d\mathbf{p}' + \sum_b \sum_{i,j=1}^5 f_a^0(\mathbf{p}) \Psi_i^a(\mathbf{p}) \langle \Psi_i^a | \delta\hat{I} | \Psi_j^b \rangle \int \Psi_j^b(\mathbf{p}') \delta f_b(\mathbf{p}') d\mathbf{p}', \quad (61)$$

where $f_a^0(\mathbf{p})$ is the local equilibrium distribution function (with different temperatures and mean velocities), ν_a is the inverse time of the heat flux relaxation of component a , and $\langle \Psi_i^a | \delta\hat{I} | \Psi_j^b \rangle$ represents the matrix elements of the linearized collision integral of the Balescu-Lenard integral, for example [24]:

$$\begin{aligned} \langle \Psi_i^a | \delta\hat{I} | \Psi_j^b \rangle = & -\frac{2}{n_a} \sum_c \int d\mathbf{k} d\mathbf{p}_1 d\mathbf{p}_2 \frac{e_a^2 e_c^2}{k^4 |\varepsilon(\mathbf{k}\mathbf{v}_1, \mathbf{k})|^2} \delta(\mathbf{k}\mathbf{v}_1 - \mathbf{k}\mathbf{v}_2) f_a^0(\mathbf{p}_1) f_b^0(\mathbf{p}_2) \\ & \times \mathbf{k} \frac{\partial}{\partial \mathbf{p}_1} \Psi_i^a [\delta_{ab} \mathbf{k} \frac{\partial}{\partial \mathbf{p}_1} \Psi_j^b - \delta_{bc} \mathbf{k} \frac{\partial}{\partial \mathbf{p}_2} \Psi_j^c + \left(\frac{\mathbf{k}\delta\mathbf{v}_2}{T_c} - \frac{\mathbf{k}\delta\mathbf{v}_1}{T_a} \right) (\delta_{ab} \Psi_j^b(\mathbf{p}_1) + \delta_{bc} \Psi_j^c(\mathbf{p}_2))]. \end{aligned} \quad (62)$$

In order to recover the form of the model collision integral from its linearized form (61) it suffices to use the conservation of the number of elastically interacting particles. This property, as well as the total momentum and energy conservation, is valid for both mean and fluctuating quantities. Consequently, the expression for Langevin's source intensity in the kinetic equation for the fluctuation of the distribution function [30] should satisfy the conditions

$$\sum_b \int \Psi^b(\mathbf{p}_2) (y_a y_b)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} d\mathbf{p}_2 = 0 \quad (63)$$

for $\Psi^b(\mathbf{p}_2) = 1$, \mathbf{p}_2 , $p_2^2/2m_b$.

The spectral function of Langevin's source in a non-equilibrium state is given [6–8] by following form:

$$(y_a y_b)_{\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} = -(\delta\hat{I}_{a\mathbf{p}_1} + \delta\hat{I}_{b\mathbf{p}_2}) \delta_{ab} \delta(\mathbf{p}_1 - \mathbf{p}_2) f_a(\mathbf{p}_1) + \delta_{ab} \delta(\mathbf{p}_1 - \mathbf{p}_2) I_a(\mathbf{p}_1) + I_{ab}(\mathbf{p}_1, \mathbf{p}_2), \quad (64)$$

where $I_{ab}(\mathbf{p}_1, \mathbf{p}_2)$ is the so-called “not integrated” collision operator [7]:

$$\sum_b \int I_{ab}(\mathbf{p}_1, \mathbf{p}_2) d\mathbf{p}_2 = I_a(\mathbf{p}_1). \quad (65)$$

In the case of many-component plasma this “not integrated” collision operator has the form:

$$I_{ab}(\mathbf{p}_1, \mathbf{p}_2) = 2e_a^2 e_b^2 n_b \left(\frac{\partial}{\partial p_{1i}} - \frac{\partial}{\partial p_{2i}} \right) \int \frac{k_i k_j \delta(\mathbf{k}\mathbf{v}_1 - \mathbf{k}\mathbf{v}_2)}{k^4 |\varepsilon(\mathbf{k}\mathbf{v}_1, \mathbf{k})|^2} \left(\frac{\partial}{\partial p_{1j}} - \frac{\partial}{\partial p_{2j}} \right) f_a f_b d\mathbf{k}. \quad (66)$$

Summing (64) and (65) over b and integrating over \mathbf{p}_2 , and taking into account (63), we get

$$I_a(\mathbf{p}_1) = \frac{1}{2} \sum_b \int (\delta \hat{I}_{a\mathbf{p}_1} + \delta \hat{I}_{b\mathbf{p}_2}) \delta_{ab} \delta(\mathbf{p}_1 - \mathbf{p}_2) f_a(\mathbf{p}_1) d\mathbf{p}_2. \quad (67)$$

Since (63) and (64) are of a general character, the relation (67) is valid both for “exact” and model collision integrals. From (61) and (67), we have:

$$I_a(\mathbf{p}) = \frac{1}{2} \sum_b \sum_{i=1}^5 f_a^0(\mathbf{p}) \Psi_i^a(\mathbf{p}) \langle \Psi_i^a | \delta \hat{I} | \Psi_1^b \rangle. \quad (68)$$

The matrix elements $\langle \Psi_i^a | \delta \hat{I} | \Psi_1^b \rangle$ can be calculated for example for Coulomb plasma

$$\langle \Psi_i^a | \delta \hat{I} | \Psi_1^b \rangle = - \sum_{r=1}^3 \delta_{ir+1} \sum_c \nu_{ac} \frac{T_c V_{ra} - T_a V_{rc}}{T_a T_c} \sqrt{\frac{m_a}{T_a}} T_{ac} (\delta_{ab} + \delta_{bc}) - \delta_{i5} \sum_c \nu_{ac} \sqrt{6} \frac{T_a - T_b}{T_a} \frac{\mu_{ac}}{m_c} (\delta_{ab} + \delta_{bc}), \quad (69)$$

where ν_{ab} is the momentum relaxation frequency for plasma:

$$\nu_{ab} = \frac{4}{3} \sqrt{2\pi} \frac{e_a^2 e_b^2 n_b \mu_{ab}}{m_a^{1/2} m_b^{3/2} T_{ab}^{3/2}} \ln 1/\lambda \quad (70)$$

and

$$\mu_{ac} = \frac{m_a m_c}{m_a + m_c}; T_{ac} = \frac{m_a T_c + m_c T_a}{m_a + m_c}$$

Substituting (69) into (68), we obtain a quite simple and, at the same time, sufficiently rigorous form of the model collision integral for many-component plasma in the local equilibrium state, which describes in the customary form the mean velocity and temperature relaxation:

$$I_a(\mathbf{p}) = - \sum_b \nu_{ab} f_a^0(\mathbf{p}) \left[\delta \mathbf{v}_a m_a \frac{\mathbf{V}_a - \mathbf{V}_b}{T_a} + \left(\frac{m_a}{T_a} \delta \mathbf{v}_a^2 - 3 \right) (T_a - T_b) \frac{m_a}{m_a + m_b} \right]. \quad (71)$$

The first term in (71) describes relaxation of the momenta (we took into account the isothermal case) and the second term describes the temperature relaxation (we neglected corrections due to the square mean velocities). For this form of the collision integral (71) it is easy to verify the Boltzmann H-theorem:

$$\frac{\partial}{\partial t} H(t) = \frac{\partial}{\partial t} \sum_a \int f_a(\mathbf{p}, t) \ln f_a(\mathbf{p}, t) d\mathbf{p} = - \sum_a \int \frac{\delta \mathbf{p}^2}{2m_a T_a} I_a(\mathbf{p}) d\mathbf{p} = - \frac{3}{4} \sum_{ab} \nu_{ab} \frac{m_a n_a}{m_a + m_b} \frac{(T_a - T_b)^2}{T_a T_b} \leq 0. \quad (72)$$

Finally we can combine (71) with (55)

$$I_a(\mathbf{p}) = -\nu_a \left\{ f_a(\mathbf{p}) - f_a^0(\mathbf{p}) \left[1 - \left(\frac{1 - \text{Pr}}{\text{Pr}} \right) P_{aij} \frac{\delta v_{ai} \delta v_{aj}}{2P_a T_a} m_a \right] \right\} - \sum_b \nu_{ab} f_a^0(\mathbf{p}) \delta \mathbf{v}_a m_a \frac{\mathbf{V}_a - \mathbf{V}_b}{T_a} - \sum_b \nu_{ab} f_a^0(\mathbf{p}) \left(\frac{m_a}{T_a} \delta \mathbf{v}_a^2 - 3 \right) (T_a - T_b) \frac{m_a}{m_a + m_b}. \quad (73)$$

The time evolution of the many-component systems up to the hydrodynamic stage can be described as follows: a first, local equilibrium components achieves, and then a balance across the gas mixture comes. At all these stages Boltzmann's H-theorem holds. Thus, the complicated exponential dependence typical of the GK model appears to be unfounded and does not hold for states remote from the full equilibrium.

III. CONCLUSION

Using the well-known projection technique, a new form of the collision operator for the Boltzmann gas of hard spheres and for the Coulomb plasma has been developed. The proposed collision operator takes into account relaxation of the first 13 hydrodynamic moments properly and accounts for the contribution of non-diagonal components in the expansion of the linearized collision operator in the complete system of Hermite polynomials. The non-diagonal components accounted for in this basis in the quadratic approximation contribute to the diagonal components. It is shown that for a system of charged particles with the Coulomb interaction potential, these contributions are essential and comparable with Spitzer corrections to the transport coefficients. In the case of the Boltzmann gas of hard spheres these corrections are insignificant. In the case of a many-component system, the nonlinear model collision integral is constructed on the basis of the linearized one. Unlike previous cases, it does not exhibit any complicated exponential dependence and avoids the coefficient ambiguity in the many-component collision integral. Boltzmann's H-theorem is proved for our model.

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